JOURNAL OF APPROXIMATION THEORY 44, 55–68 (1985)

Constructive Methods for Fourth-Order Elliptic Equations*

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Communicated by Oved Shisha

Received September 26, 1983

I. INTRODUCTION

In this paper we shall develop constructive methods for solving the fourth-order elliptic equation

$$\Delta\Delta u + au_{xx} - 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0. \qquad (1.1)$$

Equations of this type occur frequently in the mathematical theory of elasticity. For example, the differential equation of bending of an isotropic shell, subject to tensile forces in the middle plane and lying on an elastic foundation, is given by [3]

$$\frac{Eh^2}{12(1-\mu^2)}\Delta\Delta u - H_x\frac{\partial^2 u}{\partial x^2} - Hy\frac{\partial^2 u}{\partial y^2} + c_1 u - c_2\Delta u = q(x, y). \qquad (1.2)$$

Here u is taken to be the vertical displacement.

There are two well-known function theoretic approaches to the solution of this problem (see Bergman [1, 2] and Vekua [4]). Both of these methods reformulate Eq. (1.1) as a complex hyperbolic equation and then

* This work was supported in part by the Department of Energy through research grant DE-AC01-81ER10967.

0021-9045/85 $3.00

seek an integral representation in terms of a kernel which is to be determined. If the complex coordinates $z := x + iy$, and $\zeta := x - iy$ are introduced, the equation (1.1) becomes formally hyperbolic, i.e.,

$$\mathbf{L}[U] := U_{zz\zeta\zeta} + MU_{zz} + LU_{z\zeta} + NU_{\zeta\zeta} + AU_z + DU_\zeta + CU = 0, \quad (1.3)$$

where the new coefficients are

$$M = \bar{N} = \tfrac{1}{16}[a - c + ib], \qquad L = \tfrac{1}{8}[a + c], \qquad A = \bar{D} = \tfrac{1}{16}(d + ie), \quad (1.4)$$

and $C = \frac{1}{16}f$.

It is of course understood that the coefficients of (1.1) are assumed to be analytic function of x and y in a sufficiently large bicyllinder so that our continuation to the z, ζ space is meaningful. For simplicity of discussion let us assume that the coefficients are entire.

Bergman [2] has shown that all solutions of (1.3), in a neighborhood of the origin, may be represented in the form

$$\begin{aligned} U(z, \zeta) = \sum_{k=1}^{2} \int_{-1}^{1} \bigg[& E^{(\mathrm{I},k)}(z, \zeta, t) f_k \left(\frac{1}{2} z[1 - t^2] \right) \\ & + E^{(\mathrm{II},k)}(z, \zeta, t) g_k \left(\frac{1}{2} \zeta[1 - t^2] \right) \bigg] \frac{dt}{\sqrt{1 - t^2}}, \end{aligned} \quad (1.5)$$

where the functions $E^{(x,k)}(z, \zeta, t)$ satisfy a partial differential equation (to be specified below), and the initial conditions

$$\begin{aligned} E^{(\mathrm{I},1)}(z, 0, t) &= E^{(\mathrm{II},1)}(0, \zeta, t) = 1, \\ E^{(\mathrm{I},2)}(z, 0, t) &= E^{(\mathrm{II},2)}(0, \zeta, t) = 0 \\ E_\zeta^{(\mathrm{I},1)}(z, 0, t) &= E_z^{(\mathrm{II},1)}(0, \zeta, t) = 0 \\ E_\zeta^{(\mathrm{I},2)}(z, 0, t) &= E_z^{(\mathrm{II},2)}(0, \zeta, t) = 1. \end{aligned} \quad (1.6)$$

The functions $E^{(\mathrm{I},k)}$ satisfy the differential equation [2]

$$\begin{aligned} & z^{-1}t^{-1}(1 - t^2)[E_{z\zeta\zeta t} + ME_{zt} + \tfrac{1}{2}LE_{t\zeta} + \tfrac{1}{2}AE_t] \\ & \quad + \tfrac{1}{4}z^{-2}t^{-2}(1 - t^2)^2[E_{\zeta\zeta tt} + ME_{tt}] - z^{-1}t^{-2}[E_{z\zeta\zeta} + ME_z + \tfrac{1}{2}LE_\zeta \\ & \quad + \tfrac{1}{2}AE] - \tfrac{3}{4}z^{-2}t^{-3}(1 - t^4)[E_{t\zeta\zeta} + ME_t] + \tfrac{3}{4}z^{-2}t^{-4}[E_{\zeta\zeta} + ME] \\ & \quad + L[E] = 0, \end{aligned} \quad (1.7)$$

and the functions $E^{(\mathrm{II},k)}$ satisfy the differential equation

$$\begin{aligned}&\zeta^{-1}t^{-1}(1-t^2)[E_{\zeta zzt}+NE_{\zeta t}+\tfrac{1}{2}LE_{tz}+\tfrac{1}{2}DE_t]\\&\quad+\tfrac{1}{4}\zeta^{-2}t^{-2}(1-t^2)^2[Z_{zztt}+NE_{tt}]-z^{-1}t^{-2}[E_{\zeta zz}+NE_\zeta\\&\quad+\tfrac{1}{2}LE_z+\tfrac{1}{2}DE]-\tfrac{3}{4}\zeta^{-2}t^{-3}(1-t^4)[E_{tzz}+NE_t]\\&\quad+\tfrac{3}{4}\zeta^{-2}t^{-4}[E_{zz}+NE]+\mathbf{L}^*(E)=0\end{aligned}\tag{1.7'}$$

where

$$\mathbf{L}^*(E)=E_{\zeta\zeta zz}+\bar{M}E_{\zeta\zeta}+LE_{\zeta z}+\bar{N}E_{zz}+\bar{A}E_\zeta+DE_z+CE.$$

However, what is more significant is that Bergman has given a recursive scheme for computing the coefficients $P^{(\nu)}(z,\zeta)$ of the Taylor expansion of E as a function of t; namely, if

$$E^{(\mathrm{I})}(z,\zeta,t):=P^{(0)}(z,\zeta)+\sum_{\nu=1}^{\infty}t^{2\nu}z^\nu P^{(\nu)}(z,\zeta),\tag{1.8}$$

then these coefficients may be computed by solving recursively the differential equations

$$\begin{aligned}&\mathbf{D}_1(P^{(0)})=0,\\&\mathbf{D}_1(P^{(1)})=-4\mathbf{D}_1(P_z^{(0)})-2\mathbf{D}_2(P^{(0)}),\\&\mathbf{D}_1(P^{(n+2)})=-\frac{1}{n^2+2n+3/4}\Bigg[\mathbf{D}_1(P_{zz}^{(n)})+(2n+1)\,\mathbf{D}_1(P_z^{(n+1)})\\&\qquad+\mathbf{D}_2(P_z^{(n)})+\left(n+\frac{1}{2}\right)\mathbf{D}_2(P^{(n+1)})+NP_{\zeta\zeta}^{(n)}+DP_\zeta^{(n)}+CP^{(n)}\Bigg],\\&\qquad n\geqslant 0.\end{aligned}\tag{1.9}$$

Here $\mathbf{D}_1(H):=H_{\zeta\zeta}+MH$ and $\mathbf{D}_2(H):=LH_\zeta+AH$.

It has been shown [2] that there exist two sequences of functions $P^{(\mathrm{I},1,n)}(z,\zeta)$ and $P^{(\mathrm{I},2,n)}(z,\zeta)$, $n\geqslant 0$, which satisfying (1.9) such that the initial conditions

$$\begin{aligned}&P^{(\mathrm{I},1,0)}(z,0)=1, && P_\zeta^{(\mathrm{I},1,0)}(z,0)=0,\\&P^{(\mathrm{I},1,n)}(z,0)=0, && P_\zeta^{(\mathrm{I},1,n)}(z,0)=0 \quad (n=1,2,\ldots),\\&P^{(\mathrm{I},2,0)}(z,0)=0, && P_\zeta^{(\mathrm{I},2,0)}(z,0)=1,\\&P^{(\mathrm{I},2,n)}(z,0)=0, && P_\zeta^{(\mathrm{I},2,n)}(z,0)=0 \quad (n=1,2,\ldots).\end{aligned}\tag{1.10}$$

Similarly, we may get the Taylor expansion of $E^{(\mathrm{II})}$ as a function of t.

$$E^{(\mathrm{II})}(z, \zeta, t) = P^{(\mathrm{II},0)}(z, \zeta) + \sum_{\nu=1}^{\infty} t^{2\nu} z^{\nu} P^{(\nu)}(z, 3) \tag{1.8'}$$

their coefficients may be computed by solving recursively the differential equations

$$\begin{aligned} \mathbf{D}_1^*(P^{(\mathrm{II},0)}) &= 0 \\ \mathbf{D}_1^*(P^{(\mathrm{II},1)}) &= -4\mathbf{D}_1^*(P_\zeta^{(\mathrm{II},0)}) - 2\mathbf{D}_2^*(P^{(\mathrm{II},0)}) \\ \mathbf{D}_1^*(P^{(\mathrm{II},n+2)}) &= -\frac{1}{n^2+2n+3/4}\Big[\mathbf{D}_1^*(P_{\zeta\zeta}^{(\mathrm{II},n)}) + (2n+1)\,\mathbf{D}_1^*(P_\zeta^{(\mathrm{II},n+1)}) \\ &\quad + \mathbf{D}_2^*(P_\zeta^{(\mathrm{II},n)}) + \left(n+\frac{1}{2}\right)\mathbf{D}_2^*(P^{(\mathrm{II},n+1)}) \\ &\quad + MP_{zz}^{(\mathrm{II},n)} + AP_z^{(\mathrm{II},n)} + CP^{(\mathrm{II},n)}\Big] \qquad (n \geqslant 0) \end{aligned} \tag{1.9'}$$

Here $\mathbf{D}_1^*(H) := H_{zz} + NH$, $\mathbf{D}_2^* := LH_z + BH$.

Another representation has been given to the complex equation (1.3) by Vekua [4]. This representation makes use of the complex Riemann function. In this paper we shall show that these two methods are equivalent, and moreover, use the Bergman recursive scheme to compute the Riemann function. This will lead to a new representation for the solution of the Goursat problem, and several examples illustrating special cases of interest will be worked out.

II. Goursat's Problem

As the solutions to (1.3) are uniquely determined by their Goursat data, the solutions generated by the Bergman and Vekua mappings may be put in a one-to-one correspondence via their respective Goursat data. Moreover, the analytic associates of each of these mappings may also be put in a one-to-one correspondence. This in turn will show the equivalence of these two approaches. To this end let us introduce the functions

$$\begin{aligned} \phi_k(z) &:= \int_{-1}^{1} f_k\left(\frac{z}{2}[1-t^2]\right)\frac{dt}{\sqrt{1-t^2}}, \\ \psi_k(\zeta) &:= \int_{-1}^{1} g_k\left(\frac{\zeta}{2}[1-t^2]\right)\frac{dt}{\sqrt{1-t^2}} \qquad (k=1,2), \end{aligned} \tag{2.1}$$

where f_k and g_k are the analytic functions of the representation (1.5). Then we have the identities

$$z^n \int_{-1}^{1} t^{2n} f_k\left(\frac{z}{2}[1-t^2]\right)\frac{dt}{\sqrt{1-t^2}} = \frac{1}{2^{2n}B(n,n+1)}\int_0^z (z-s)^{n-1}\,\phi_k(s)\,ds,$$

$$\zeta^n \int_{-1}^{1} t^{2n} g^k\left(\frac{\zeta}{2}[1-t^2]\right)\frac{dt}{\sqrt{1-t^2}} = \frac{1}{2^{2n}B(n,n+1)}\int_0^\zeta (\zeta-\sigma)^{n-1}\,\psi_k(\sigma)\,d\sigma$$

$(n=1,2,\ldots)$. and the representation (1.5), (1.8) may be written in the form

$$\begin{aligned} U(z,\zeta) = \sum_{k=1}^{2}&\left[P^{(\mathrm{I},k,0)}(z,\zeta)\,\phi_k(z) + \sum_{n=1}^{\infty}\frac{P^{(\mathrm{I},k,n)}(z,\zeta)}{2^{2n}B(n,n+1)}\int_0^z (z-s)^{n-1}\right.\\ &\left.\cdot\,\phi_k(s)\,ds\right] + \left[P^{(\mathrm{II},k,0)}(z,\zeta)\,\psi_k(\zeta) + \sum^{\infty} P^{(\mathrm{II},k,n)}(z,\zeta)\right. \qquad (2.2)\\ &\left.\cdot\int_0^\zeta (\zeta-\sigma)^{n-1}\,\psi_k(\sigma)\,d\sigma\right] \end{aligned}$$

If the origin is shifted from (0.0) in the $z-\zeta$ space to (t,τ), the representation (2.2) takes the form

$$\begin{aligned} U(z,\zeta) = \sum_{k=1}^{2}&\left\{\left[P^{(\mathrm{I},k,0)}(z,\zeta,\tau)\,\phi_k(z) + \sum_{n=1}^{\infty}\frac{P^{(\mathrm{I},k,n)}(z,\zeta,\tau)}{2^{2n}B(n,n+1)}\right.\right.\\ &\left.\cdot\int_t^z (z-s)^{n-1}\,\phi_k(s)\,ds\right] + \left[P^{(\mathrm{II},k,0)}(z,\zeta,t)\,\psi_k(\zeta)\right. \qquad (2.3)\\ &\left.\left. + \sum_{n=1}^{\infty}\frac{P^{(\mathrm{II},k,n)}(z,\zeta,t)}{2^{2n}B(n,\ n+1)}\int_\tau^\zeta (\zeta-\sigma)^{n-1}\,\psi_k(\sigma)\,dr\right]\right\}, \end{aligned}$$

where the $P^{(\mathrm{I},k,0)}(z,\zeta,\tau)$ are solutions of the system (1.9) which satisfy the initial conditions

$$\begin{aligned} &P^{(\mathrm{I},1,0)}(z,\tau,\tau)=1, \qquad P_\zeta^{(\mathrm{I},1,0)}(z,\tau,\tau)=0,\\ &P^{(\mathrm{I},1,n)}(z,\tau,\tau)=0, \qquad P_\zeta^{(\mathrm{I},1,n)}(z,\tau,\tau)=0 \qquad (n\geqslant 1), \end{aligned} \qquad (2.4)$$

and

$$\begin{aligned} &P^{(\mathrm{I},2,n)}(z,\tau,\tau)=0, \qquad P_\zeta^{(\mathrm{I},2,0)}(z,\tau,\tau)=1,\\ &P^{(\mathrm{I},2,n)}(z,\tau,\tau)=0, \qquad P_\zeta^{(\mathrm{I},2,n)}(z,\tau,\tau)=0 \qquad (n\geqslant 1). \end{aligned} \qquad (2.5)$$

The $P^{(\mathrm{II},k,n)}(z, \tau, \tau)$ are likewise solutions of the system (1.9′), and moreover, satisfy

$$\begin{aligned} &P^{(\mathrm{II},1,0)}(t, \zeta, t) = 1, && P_z^{(\mathrm{II},1,0)}(t, \zeta, t) = 0, \\ &P^{(\mathrm{II},1,n)}(t, \zeta, t) = 0, && P_z^{(\mathrm{II},1,n)}(t, \zeta, t) = 0 \qquad (n \geqslant 1), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &P^{(\mathrm{II},2,0)}(t, \zeta, t) = 0, && P_z^{(\mathrm{II},2,0)}(t, \zeta, t) = 1, \\ &P^{(\mathrm{II},2,n)}(t, \zeta, t) = 0, && P_z^{(\mathrm{II},2,n)}(t, \zeta, t) = 0 \qquad (n \geqslant 1). \end{aligned} \tag{2.7}$$

The complex Riemann function, $G(t, \tau, z, \zeta)$, for Eq. (1.3) is a solution of this equation, which moreover, satisfies the special Goursat data [4] (see, in particular, pp. 186, 187, Eq. (37.9), (37.11), (37.12).)

$$\begin{aligned} &G(t, \tau, t, \zeta) = 0, && \frac{\partial G}{\partial z}(t, \tau, t, \zeta) = X(\zeta, t, \tau), \\ &G(t, \tau, z, \tau) = 0, && \frac{\partial G}{\partial \zeta}(t, \tau, z, \tau) = X^*(z, t, \tau). \end{aligned} \tag{2.8}$$

The functions X, X^* appearing in the characteristic conditions (2.8) are solutions of ordinary differential equations, which satisfy prescribed initial conditions. In our case, these are

$$\begin{aligned} &\frac{d^2X}{d\zeta^2} + MX = 0, \\ &X(\tau, t, \tau) = 0, \qquad \frac{dX}{d\zeta}(\tau, t, \tau) = 1. \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\frac{d^2X^*}{dz^2} + NX^* = 0, \\ &X^*(t, t, \tau) = 0, \qquad \frac{dX^*}{dz}(t, t, \tau) = 1. \end{aligned} \tag{2.10}$$

Having obtained the Riemann function we may use it to solve the general Goursat problem for (1.3), where

$$\begin{aligned} &U(z, \tau) = f_0(z), && U_\zeta(z, \tau) = f_1(z), \\ &U(t, \zeta) = g_0(\zeta), && U_z(t, \zeta) = g_1(\zeta), \end{aligned} \tag{2.11}$$

where the data $f_k(z)$, $g_k(\zeta)$ are required, moreover, to satisfy the consistency conditions

$$f_m^{(k)}(t) = g_k^{(m)}(\tau) \qquad (k, m = 0, 1). \tag{2.12}$$

We have shown the equivalence of Bergman's and Vekua's method; hence, we may also represent this solution using the representation (2.3). To this end, we must first solve for the analytic functions $\phi_k(z)$, $\psi_k(\zeta)$ $(k = 1, 2)$. From (2.11) and (2.3) (2.7) we obtain

$$f_0(z) = \phi_1(z) + \sum_{k=1}^{2} P^{(\mathrm{II},k,0)}(z, \tau, t)\,\psi_k(\tau), \tag{2.13}$$

$$g_0(\zeta) = \psi_1(\zeta) + \sum_{k=1}^{2} P^{(\mathrm{I},k,0)}(t, \zeta, \tau)\,\phi_k(t), \tag{2.14}$$

$$\begin{aligned} f_1(z) = \phi_2(z) + \sum_{k=1}^{2} [&P^{(\mathrm{II},k,0)}(z, \tau, t)\,\psi_k(\tau) + P^{(\mathrm{II},k,0)} \\ &\cdot (z, \tau, t)\,\psi_k'(\tau) + 1/2\,P^{(\mathrm{II},k,1)}(z, \tau, t)\,\psi_k(\tau)], \end{aligned} \tag{2.15}$$

$$\begin{aligned} g_1(\zeta) = \psi_2(\zeta) + \sum_{k=1}^{2} [&P_z^{(\mathrm{I},k,0)}(t, \zeta, \tau)\,\phi_k(t) + P^{(\mathrm{I},k,0)} \\ &\cdot (t, \zeta, \tau)\,\phi_k'(t) + 1/2\,P^{(\mathrm{I},k,1)}(t, \zeta, \tau)\,\phi_k(t)]. \end{aligned} \tag{2.16}$$

Clearly the $\phi_k(t)$, $\phi_k'(t)$, $\psi_k(t)$, $\psi_k'(t)$ may be determined from consistency conditions, which we illustrate below for several special cases.

If we consider Eq. (1.1) of the form

$$\Delta^2 u + a(x, y)\,\Delta u + b(x, y)\,U_x + c(x, y)\,U_y + d(x, y)\,u = 0, \tag{2.17}$$

then the complex version of this is

$$U_{zz\zeta\zeta} + LU_{z\zeta} + AU_z + BU_\zeta + CU = 0. \tag{2.18}$$

In this case, the functions $X(\zeta, t, \tau)$ and $X^*(z, t, \tau)$ of (2.8)–(2.10) satisfy the simplified initial value problems

$$\frac{d^2X}{d\zeta^2} = 0, \qquad X = 0 \quad \text{at } \zeta = \tau, \qquad \frac{dX}{d\zeta} = 1 \quad \text{at } \zeta = \tau, \tag{2.19}$$

and

$$\frac{d^2X^*}{dz^2} = 0, \qquad X^* = 0 \quad \text{at } z = t, \qquad \frac{dX^*}{dz} = 1 \quad \text{at } z = t;$$

hence

$$X(\zeta, t, \tau) = \zeta - \tau, \text{ and } X^*(z, t, \tau) = z - t. \tag{2.20}$$

The operators $\mathbf{D}_1$ and $\mathbf{D}_2$, moreover, become

$$\mathbf{D}_1(H) = H_{\zeta\zeta}, \text{ and } \mathbf{D}_2(H) := LH_\zeta + AH, \tag{2.21}$$

which permits us to compute, using (1.9),

$$P^{(\mathrm{I},1,0)}(z, \zeta, \tau) = 1 \tag{2.22}$$

$$P^{(\mathrm{I},1,1)}(z, \zeta, \tau) = -2 \int_\tau^\zeta d\sigma \int_t^\sigma ds \; A(z, s), \tag{2.23}$$

and

$$P^{(\mathrm{I},2,1)}(z, \zeta, \tau) = -2 \int_\tau^\zeta d\sigma \int_\tau^\sigma ds \; [L(z, s) + A(z, s)(s-t)]. \tag{2.24}$$

Likewise,

$$\mathbf{D}_1^*(H) = H_{zz} \qquad \text{and} \qquad \mathbf{D}_2^*(H) = LH_z + BH; \tag{2.25}$$

hence,

$$P^{(\mathrm{II},1,0)}(\zeta, z, t) \equiv 1, \tag{2.26}$$

$$P^{(\mathrm{II},1,1)}(\zeta, z, t) = -2 \int_t^z d\sigma \int_t^\sigma B(s, \zeta)\, ds, \tag{2.27}$$

and

$$P^{(\mathrm{II},2,1)}(\zeta, z, t) = -2 \int_t^z d\sigma \int_t^\sigma ds[L(s, \zeta) + B(s, \zeta)(s-t)]. \tag{2.28}$$

Putting these coefficients into the equations for the Goursat data (2.13)–(2.16), that is for

$$f_0(z) = g_0(\zeta) = 0, \qquad g_1(z) = X(\zeta, t, \tau), \qquad f_1(\zeta) = X^*(z, t, \tau),$$

yields

$$\phi_1(t) + (\zeta - \tau)\, \phi_2(t) + \psi_1(\zeta) = 0, \tag{2.29}$$

$$\phi_1(z) + \psi_1(\tau) + (z-t)\, \psi_2(\tau) = 0, \tag{2.30}$$

$$\phi_1'(t)+(\zeta-\tau)\,\phi_2'(t)+\psi_2(\zeta)-\phi_1(t)\int_\tau^\zeta d\tau\int_\tau^\sigma A(t,s)\,ds$$

$$-\phi_2(t)\int_\tau^\zeta d\tau\int_\tau^\sigma [L(t,s)+A(t,s)(s-\tau)]\,ds$$

$$=X(\zeta,t,\tau):=\zeta-t, \tag{2.31}$$

$$\phi_2(z)+\psi_1'(\tau)+(z-t)\,\psi_2'(\tau)-\psi_1(\tau)\int_t^z d\sigma\int_t^\sigma B(s,\tau)\,ds$$

$$-\psi_2(\tau)\int_t^z da\int_t^\sigma [L(s,\tau)+B(s,\tau)(s-\tau)(s-r)]\,ds$$

$$=X^*(z,t,\tau):=z-t. \tag{2.32}$$

The consistency condition (2.12) permit us to compute from (2.29), (2.30) that $\phi_1(t)+\psi_1(\tau)=0$, from (2.30), (2.32) that $\phi_1'(t)+\psi_2(\tau)=0$, from (2.29), (2.31) that $\phi_2(t)+\psi_1'(\tau)=0$, and from (2.31), (2.32) that $\phi_2'(t)+\psi_2'(\tau)=1$. Normalizing these otherwise arbitrary coefficients provides us with the conditions

$$\begin{aligned}
&\phi_1(t)=\phi_1'(t)=0, \qquad \phi_2(t)=0, \qquad \phi_2'(t)=1/2,\\
&\psi_1(\tau)=\psi_1'(\tau)=0, \qquad \psi_2(\tau)=0, \qquad \psi_2'(\tau)=1/2.
\end{aligned} \tag{2.33}$$

Putting these values into (2.29)–(2.32) yields

$$\begin{aligned}
&\phi_1(z)\equiv 0, \qquad \psi_1(\zeta)\equiv 0,\\
&\phi_2(z)=\tfrac{1}{2}(z-t), \qquad \psi_2(\zeta)=\tfrac{1}{2}(\zeta-\tau).
\end{aligned} \tag{2.34}$$

Consequently, the Bergman representation provides us with a series development for the Riemann function, namely

$$\begin{aligned}
&G(t,\tau;z,\zeta)\\
&\quad=(z-t)(\zeta-\tau)+\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{P^{(\mathrm{I},2,n)}(z,\zeta,\tau)}{2^{2n}B(n,n+1)}\right.\\
&\qquad\left.\cdot\int_t^z (z-s)^{n-1}(s-t)\,ds+P^{(\mathrm{II},2,n)}(z,\zeta,t)\cdot\int^\zeta (\zeta-\sigma)^{n-1}(\sigma-\tau)\,d\sigma\right]\\
&\quad=(z-t)(\zeta-\tau)+\frac{1}{2}\sum_{n=1}^{\infty}\\
&\qquad\cdot\left[\frac{P^{(\mathrm{I},2,n)}(z,\zeta,\tau)(z-t)^{n+1}}{2^{2n}B(n,n+1)\,n(n+1)}+\frac{P^{(\mathrm{II},2,n)}(z,\zeta,t)(\zeta-\tau)^{n+1}}{2^{2n}B(n,n+1)\,n(n+1)}\right],
\end{aligned} \tag{2.35}$$

The general Goursat problem may be set up in the same way as we just did for the Riemann function. Setting the coefficients (2.22)–)2.28) into the equations for the Goursat data (2.13)–(2.16) we obtain after normalization

$$\begin{aligned}\phi_1(z) &= f_0(z) + \psi_1(\tau) - (z-t)\,\psi_2(\tau),\\ \psi_1(\zeta) &= g_0(\zeta) + \phi_1(t) - (\zeta-\tau)\,\phi_2(t).\end{aligned} \tag{2.36}$$

Likewise we have

$$\phi_2(z) = f_1(z) - \psi_1'(\tau) - (z-t)\,\psi_2'(\tau)$$
$$+ \int_t^z d\sigma \int_t^\sigma ds[\psi_1(\tau)\,B(s,\tau) + \psi_2(\tau)(L(s,\tau) + B(s,\tau)(s-t))]$$

and

$$\psi_2(\zeta) = g_1(\zeta) - \phi_1'(t) - (\zeta-\tau)\,\phi_2'(t) + \int_\tau^\zeta d\sigma \int_\tau^\sigma ds[\phi_1(t)\,A(t,s)$$
$$+ \phi_2(t)(L(t,s) + A(t,s)(s-t))]. \tag{2.37}$$

The consistency conditions plus normalization give us

$$\begin{aligned}\phi_2(t) &= \tfrac{1}{2} f_1(t) = \psi_1(\tau) = \tfrac{1}{2} g_0'(\tau), \qquad \psi_2(\tau) = \tfrac{1}{2} g_1(\tau)\\ &= \tfrac{1}{2} f_0'(t) = \phi_1'(t), \qquad \phi_2'(t) = \tfrac{1}{2} f_1'(t) = \psi_2'(\tau)\\ &= \tfrac{1}{2} g_1'(\tau).\end{aligned}$$

Hence, from (2.36), (2.37) we have

$$\begin{aligned}\phi_1(z) &= f_0(z) - \tfrac{1}{2} g_0(\tau) - \tfrac{1}{2}(z-t)\,g_1(\tau),\\ \psi_1(z) &= g_0(z) - \tfrac{1}{2} f_0(t) - \tfrac{1}{2}(z-\tau)\,f_1(t).\end{aligned} \tag{2.38}$$

$$\begin{aligned}\phi_2(z) &= f_1(z) - \tfrac{1}{2} g_0'(\tau) - \tfrac{1}{2}(z-t)\,g_1'(\tau)\\ &\quad + \frac{1}{2}\int_t^z d\sigma \int_t^\sigma [g_0(\tau)\,B(s,\tau) + g_1(\tau)(L(s,\tau)\\ &\quad + (s-t)\,B(s,\tau))]\,ds,\end{aligned} \tag{2,39}$$

and

$$\psi_2(\zeta) = g_1(\zeta) - \tfrac{1}{2} f_0'(t) - \tfrac{1}{2}(\zeta-\tau)\,f_1'(t)$$
$$+ \frac{1}{2}\int_\tau^\zeta d\sigma \left[\int_\tau^\sigma f_0(t)\,A(t,s) + f_1(t)(L(t,s) + (s-t)\,A(t,s))\right] ds.$$

Using these expressions for the functions $\phi_k(z)$ $(k=1,2)$, $\psi_k(\zeta)$ $(k=1,2)$ in (2.3) leads to a representation of the Goursat problem with normalized data.

III. Constant Coefficients

In this section we consider the special case of Eq. (1.3) where all the coefficients are constants. This leads to a great simplification in our results. We determine first the few coefficients $P^{(\mathrm{I},\nu,n)}$. As

$$\mathbf{D}_1 P^{(\mathrm{I},1,0)} := P^{(\mathrm{I},1,0)}_{\zeta\zeta} + MP^{(\mathrm{I},1,0)} = 0$$
$$P^{(\mathrm{I},1,0)}(z,\tau,\tau) = 1, \qquad P^{(\mathrm{I},1,0)}_{\zeta}(z,\tau,\tau) = 0,$$

we have

$$P^{(\mathrm{I},1,0)}(z,\ \zeta,\ \tau) = \cosh \lambda(\zeta - t), \tag{3.1}$$

where $\lambda^2 := -M \neq 0$.
The coefficient $P^{(\mathrm{I},1,1)}$ may be computed from the scheme (2.4) to be

$$P^{(\mathrm{I},1,1)}(z,\zeta,\tau) = \frac{L - A(\zeta - \tau)}{\lambda} \sinh \lambda(\zeta - \tau) - L\zeta \cosh \lambda(\zeta - \tau). \tag{3.2}$$

Likewise, using (2.5) we have

$$P^{(\mathrm{I},2,0)}(z,\ \zeta,\ t) = \frac{1}{\lambda} \sinh(\zeta - \tau), \tag{3.3}$$

and

$$P^{(\mathrm{I},2,1)}(z,\zeta,\tau) = \frac{A}{\lambda^3} \sinh \lambda(\zeta - \tau) - (\zeta - \tau)\left[\frac{L}{\lambda} \sinh \lambda(\zeta - \tau) + \frac{A}{\lambda^2} \cosh \lambda(\zeta - \tau)\right]. \tag{3.4}$$

The coefficients $P^{(\mathrm{II},\nu,n)}$ are derived in a similar manner starting with

$$\mathbf{D}_1^* P^{(\mathrm{II},1,0)}(z,\zeta,t) := P^{(\mathrm{II},1,0)}_{zz} - \mu^2 P^{(\mathrm{II},1,0)} = 0, \qquad -\mu^2 := N,$$
$$P^{(\mathrm{II},1,0)}(t,\zeta,t) = 1, \qquad P^{(\mathrm{II},1,0)}_{z}(t,\zeta,t) = 0.$$

We obtain

$$P^{(\mathrm{II},1,0)}(z,\zeta,t) = \cosh \mu(z - t), \tag{3.5}$$

$$P^{(\mathrm{II},2,0)}(z,\zeta,t) = \frac{1}{\mu} \sinh \mu(z - t), \tag{3.6}$$

$$P^{(\mathrm{II},2,1)}(t,\zeta,t) = \frac{B}{\mu^3} \sinh \mu(z - t) - (z - t)\left[\frac{L}{\mu} \sinh \mu(z - t) + \frac{B}{\mu^2} \cosh \mu(z - t)\right]. \tag{3.7}$$

We now turn to determining the Riemann function for this case. The function $X(\zeta, t, \tau)$ satisfies

$$\frac{d^2X}{d\zeta^2} - \lambda^2 X = 0, \qquad X(\tau, t, \tau) = 0, \qquad \left.\frac{dX}{d\zeta}\right|_{\zeta=\tau} = 1;$$

hence, we have

$$X(\zeta, t, \tau) = \frac{1}{\lambda} \sinh \lambda(\zeta - \tau). \tag{3.8}$$

In a similar manner we find X^* to be

$$X^*(z, t, \tau) = \frac{1}{\mu} \sinh \mu(z - t). \tag{3.9}$$

The Goursat data which the Riemann function satisfies leads to the following equations for the auxillary function ϕ_k, ψ_k, $k = 1.2$:

$$0 = \phi_1(z) + \sum_{k=1}^{2} P^{(\mathrm{II},k,0)}(z, \tau, t)\, \psi_k(\tau) = \phi_1(z) + \psi_1(\tau) \cosh \mu(z - t) + \frac{1}{\mu^2} \psi_2(\tau) \sinh \mu(z - t), \tag{3.10}$$

$$0 = \psi_1(\zeta) + \sum_{k=1}^{2} P^{(\mathrm{I},k,0)}(t, \zeta, \tau)\, \phi_k(t) = \psi_1(\zeta) + \phi_1(t) \cosh \lambda(\zeta - \tau) + \frac{1}{\lambda} \phi_2(t) \sinh \lambda(\zeta - \tau), \tag{3.11}$$

$$\begin{aligned}\frac{1}{\mu} \sinh \mu(z - t) = {}& \phi_2(z) + \psi_1'(\tau) \cosh \mu(z - t) + \frac{1}{\mu} \psi_2'(\tau) \sinh \mu(z - t) \\ & + \frac{1}{2} \psi_1(\tau) \left[\frac{L}{\mu} \sinh \mu(z - t) - (z - t) \right. \\ & \left. \times \left(\frac{B}{\mu} \sinh \mu(z - t) + L \cosh \mu(z - t) \right) \right] \\ & + \frac{1}{2} \psi_2(\tau) \left[\frac{B}{\mu^3} \sinh \mu(z - t) + (z - t) \right. \\ & \left. \times \left(\frac{L}{\mu} \sinh \mu(z - t) + \frac{B}{\mu^2} \cosh(z - t) \right) \right],\end{aligned} \tag{3.12}$$

and

$$\frac{1}{\lambda}\sinh\lambda(\zeta-\tau)=\psi_2(\zeta)+\phi_1'(t)\cosh\lambda(\zeta-\tau)+\frac{1}{\lambda}\phi_2'(t)\sinh\lambda(\zeta-\tau)$$

$$+\frac{1}{2}\phi_1(t)\left[\frac{L}{\lambda}\sinh\lambda(\zeta-\tau)-(\zeta-\tau)\left(\frac{A}{\lambda}\sinh\lambda(\zeta-\tau)\right.\right.$$

$$\left.\left.+L\cosh\lambda(\zeta-\tau)\right)\right]+\frac{1}{2}\phi_2(t)\left[\frac{A}{\lambda^3}\sinh\lambda(\zeta-\tau)+(\zeta-\tau)\right.$$

$$\left.\cdot\left(\frac{L}{\lambda}\sinh\lambda(\zeta-\tau)+\frac{A}{\lambda^2}\cosh\lambda(\zeta-\tau)\right)\right]. \tag{3.13}$$

Normalizing the auxillary functions by setting $\phi_1(t)=\psi_1(\tau)$, $\psi_2(\tau)=\phi_1'(t)$, $\phi_2(t)=\psi_1'(\tau)$, $\phi_2'(t)=\psi_2'(\tau)$ one may solve the above equations to obtain

$$\begin{aligned}&\phi_1(z)\equiv 0, \qquad \psi_1(z)\equiv 0,\\ &\phi_2(z)=\frac{1}{2\mu}\sinh\mu(z-t),\\ &\psi_2(\zeta)=\frac{1}{2\lambda}\sinh\lambda(\zeta-\tau).\end{aligned} \tag{3.14}$$

Using these in the representation (2.3) yields the following series representation for the Riemann function

$$G(t,\tau,z,\zeta)=\frac{1}{2\mu^2}\sinh^2\mu(z-t)+\frac{1}{2\mu}\sum_{n=1}^{\infty}\frac{P^{(\mathrm{I},2,n)}(z,\zeta,\tau)}{2^{2n}B(n,n+1)}$$

$$\cdot\int_\tau^z(z-s)^{n-1}\sinh\mu(s-t)\,ds+\frac{1}{2\lambda^2}\sinh^2\lambda(\zeta-\tau)$$

$$+\frac{1}{2\lambda}\sum_{n=1}^{\infty}\frac{P^{(\mathrm{II},2,n)}(z,\zeta,\tau)}{2^{2n}B(n,n+1)}\int_\tau^\zeta(\zeta-\sigma)^{n-1}$$

$$\cdot\sinh\lambda(\sigma-\tau)\,d\sigma, \tag{3.15}$$

where $\lambda^2:=-M$, $\mu^2:=-N$.

For the general Goursat problem with the data

$$\begin{aligned}&U(z,\tau)=f_0(z), \qquad U_\zeta(z,\tau)=f_1(z),\\ &U(t,\zeta)=g_0(\zeta), \qquad U_z(t,\zeta)=g_1(\zeta),\end{aligned} \tag{3.16}$$

where $f^{(m)}(t)=g^{(k)}(\tau)$, $m, k=0, 1$, a similar procedure leads to determining the associated analytic functions as

$$\phi_1(z)=f_0(z)-\frac{1}{2}g_0(\tau)\cosh\mu(z-t)-\frac{1}{2\mu}g_1(\tau)\sinh\mu(z-t),$$

$$\psi_1(s)=g_0(\zeta)-\frac{1}{2}f_0(t)\cosh\lambda(\zeta-\tau)-\frac{1}{2\mu}f_1(t)\sinh\lambda(\zeta-\tau),$$

$$\begin{aligned}\phi_2(z)=&f_1(z)-\frac{1}{2}\left(g_0'(\tau)\cosh\mu(z-t)+\frac{1}{\mu}g_1'(\tau)\sinh\mu(z-t)\right)\\ &-\frac{1}{4}g_0(\tau)\left[\frac{L}{\mu}\sinh\mu(z-t)-(z-t)\right.\\ &\left.\times\left(\frac{B}{\mu}\sinh\mu(z-t)+L\cosh\mu(z-t)\right)\right]\\ &-\frac{1}{4}g_1(\tau)\left[\frac{B}{\mu^3}\sinh\mu(z-t)-(z-t)\right.\\ &\left.\times\left(\frac{L}{\mu}\sinh\mu(z-t)+\frac{B}{\mu^2}\cosh\mu(z-t)\right)\right],\end{aligned} \tag{3.17}$$

$$\begin{aligned}\psi_2(\zeta)=&g_1(\tau)-\frac{1}{2}\left(f_0'(t)\cosh\lambda(\zeta-\tau)+\frac{1}{\lambda}f_1'(t)\sinh\lambda(\zeta-\tau)\right)\\ &-\frac{f_0(t)}{4}\left[\frac{L}{\lambda}\sinh\lambda(\zeta-\tau)-(\zeta-\tau)\right.\\ &\left.\times\left(\frac{A}{\lambda}\sinh\lambda(\zeta-\tau)+L\cosh\lambda(\zeta-\tau)\right)\right]\\ &-\frac{1}{4}f_1(t)\left[\frac{A}{\lambda^3}\sinh\lambda(\zeta-\tau)-(\zeta-\tau)\right.\\ &\left.\times\left(\frac{L}{\lambda}\sinh\lambda(\zeta-\tau)+\frac{A}{\lambda^2}\cosh\lambda(\zeta-\tau)\right)\right].\end{aligned}$$

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